THE ORDER-CONTINUOUS OPERATORS ON L_p-SPACES. PART I. GENERAL THEORY

BY
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1. **Introduction.** In this paper, a special class of operators is studied: the linear mappings of $L_\infty(S, \Sigma, \mu)$ into $L_\infty(S_0, \Sigma_0, \mu_0)$ which have bounded extensions as mappings of $L_p(S, \Sigma, \mu)$ into $L_p(S_0, \Sigma_0, \mu_0)$ for all p, $1 \leqq p \leqq \infty$. (S, Σ, μ) and (S_0, Σ_0, μ_0) are σ-finite measure spaces.

Let this class be denoted by $\mathcal{O}(\mu, \mu_0)$. It is shown that $\mathcal{O}(\mu, \mu_0)$ is a Banach space and $\mathcal{O}(\mu, \mu)$ is a starred Banach algebra having a faithful, irreducible representation on $L_2(S, \Sigma, \mu)$. Further, each $T \in \mathcal{O}(\mu, \mu_0)$, admits an "absolute value" $P_T \in \mathcal{O}(\mu, \mu_0)$, where P_T is a Banach positive operator. This implies that each $T = \mathcal{O}(\mu, \mu_0)$ maps lattice bounded intervals in $L_p(\mu)$ into lattice bounded intervals in $L_p(\mu_0)$.

Various weak and strong topologies on the unit sphere are studied, and it is shown that the extreme boundedness of these operators causes a considerable collapse of possible topologies. The real operators in $\mathcal{O}(\mu, \mu_0)$, that is, the ones which map real functions into real functions, form a complete vector lattice. Finally an important representation of these operators as set functions in a certain product space is given. In Part II, it will be shown that each of these operators induces a certain type of function-valued measure such that the action of the operator is integration with respect to the corresponding measure.

While the direction of this research has been dictated by the needs of probability theory, the author hopes that the more general setting provided here will permit these operators to find a place in other areas of mathematical research.

2. **Preliminaries and notations.** The study of $\mathcal{O}(\mu, \mu_0)$ requires a large amount of notation. Broadly speaking, the conventions of N. Dunford and J. Schwartz [5] have been followed.

Unless express mention to the contrary is made, all Banach and Hilbert spaces are complex. (S, Σ, μ) and (S_0, Σ_0, μ_0) are σ-finite measure spaces, where S and S_0 are sets, Σ and Σ_0 are σ-algebras of subsets of S and S_0 respectively, and μ and μ_0 are countably additive, positive, σ-finite measures on Σ and Σ_0 respectively. For $1 \leqq p < \infty$, $L_p(S, \Sigma, \mu)$, or simply $L_p(\mu)$, is the set of μ-equivalence classes of measurable functions, the pth powers of whose absolute values are μ-integrable. Equipped with the usual norm, $L_p(S, \Sigma, \mu)$ is a Banach space. $L_\infty(S, \Sigma, \mu)$ is the Banach space of μ-equivalence classes of μ-essentially bounded measurable functions. $M(S, \Sigma, \mu)$ is the set of μ-equivalence classes of μ a.e. finite measurable functions. $B(S, \Sigma)$ is the set of all bounded measurable functions on (S, Σ). It is a Banach space under

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supremum norm. $L_p^R(\mu)$ denotes the space of real functions in $L_p(\mu)$. In addition to being a real Banach space, $L_p^R(\mu)$ is a complete vector lattice under the usual partial ordering of μ-equivalence classes of real functions. $M^R(S, \Sigma, \mu)$ is the space of real functions in $M(S, \Sigma, \mu)$. $L_p^R(\mu) \subset M^R(\mu)$, $1 \leqq p \leqq \infty$, and $M^R(\mu)$ is a complete vector lattice such that if a set is bounded in some $L_p^R(\mu)$ lattice then its supremum is the same whether taken in the complete vector lattice $L_p^R(\mu)$ or in $M^R(\mu)$. For $M^R(\mu)$ and $L_p^R(\mu)$, $1 \leqq p \leqq \infty$, the supremum of any bounded set B is the supremum of a suitably chosen countable subset of B.

For our purposes, $(L_1(\mu))^* = L_\infty(\mu)$; and $(L_p(\mu))^* = L_q(\mu)$, where $1 < p < \infty$ and $1/p + 1/q = 1$. Let $(S, \bar{\Sigma}, \bar{\mu})$ be the completion of (S, Σ, μ) with respect to μ. Then $(L_\infty(\mu))^* = \text{ba}(S, \bar{\Sigma}, \bar{\mu})$, the space of bounded, additive, complex-valued set functions on $(S, \bar{\Sigma})$ which vanish on sets of $\bar{\mu}$-measure zero. $\text{ba}(S, \bar{\Sigma}, \bar{\mu})$ is a Banach space which contains a closed linear manifold $\text{ca}(S, \bar{\Sigma}, \bar{\mu})$, the countably additive elements in $\text{ba}(S, \bar{\Sigma}, \bar{\mu})$. By the Radon-Nikodym theorem there is an isometric isomorphism κ of $\text{ca}(S, \bar{\Sigma}, \bar{\mu})$ into $L_1(S, \Sigma, \mu)$. κ^{-1} is just the map provided by taking the indefinite μ-integrals of elements of $L_1(S, \Sigma, \mu)$. $(B(S, \Sigma))^* = \text{ba}(S, \Sigma)$, which is the set of all bounded, complex-valued, additive set functions on (S, Σ). It is a Banach space, whose real elements $\text{ba}^R(S, \Sigma)$ form a complete vector lattice. $\text{ca}(S, \Sigma)$ is a closed linear manifold of $\text{ba}(S, \Sigma)$ consisting of the countably additive elements of $\text{ba}(S, \Sigma)$. $\text{ca}^R(S, \Sigma)$ is also a complete vector lattice. It is known that $\kappa^{-1}\colon L_1^R(S, \Sigma, \mu) \subset \text{ca}^R(S, \Sigma)$ is order-preserving, and the supremum of a set bounded above in $L_1^R(S, \Sigma, \mu)$ is the same whether it is computed in the complete vector lattice $L_1^R(S, \Sigma, \mu)$ or in $\text{ca}^R(S, \Sigma)$.

$\mathscr{B}_p(\mu, \mu_0)$ denotes the set of bounded linear mappings of $L_p(\mu)$ into $L_p(\mu_0)$ for each p, $1 \leqq p \leqq \infty$. It is a Banach space, and $\mathscr{B}_p^R(\mu, \mu_0)$, the set of real operators restricted to $L_p^R(\mu)$ into $L_p^R(\mu_0)$, is a complete vector lattice. $\mathscr{B}_b(\mu, \mu_0)$ is the set of bounded linear mappings of $\text{ba}(S, \bar{\Sigma}, \mu)$ into $\text{ba}(S_0, \bar{\Sigma}_0, \bar{\mu}_0)$. If T is a linear mapping of $L_\infty(\mu)$ into $L_\infty(\mu_0)$ having bounded extensions to several different L_p-spaces, then its representative or trace for a particular p_0 is denoted T_{p_0}. To reduce notation, the norm of T_{p_0} as an element of $\mathscr{B}_{p_0}(\mu, \mu_0)$ is written simply $\|T\|_{p_0}$. If T has an extension to a bounded linear mapping of $\text{ba}(S, \bar{\Sigma}, \bar{\mu})$ into $\text{ba}(S_0, \bar{\Sigma}_0, \bar{\mu}_0)$, then its ba-trace is denoted T_b and the norm of T_b is denoted $\|T\|_b$.

A positive operator is one which carries nonnegative elements in its domain into nonnegative elements in its range. This notion of positivity induces the partial order in $\mathscr{B}_p^R(\mu, \mu_0)$: i.e., $S \geqq T$ in $\mathscr{B}_p^R(\mu, \mu_0)$ if $S - T$ is a positive operator. However, in $\mathscr{B}_2(\mu, \mu)$ there is another definition of positivity. An operator $A \in \mathscr{B}_2(\mu, \mu)$ is called positive if $(Af, f) \geqq 0$ for all $f \in L_2(\mu)$. This will be called Hilbert positivity. T^* will denote the Banach space adjoint of an operator T. $T^\#$ is the Hilbert space adjoint of an operator T. In the function spaces under discussion, $T^* = j \circ T^\# \circ j$, where j maps a function f into its complex conjugate. Clearly, when both T^* and $T^\#$ are defined, they have the same norm. If T is a positive operator, it is easy to show that $T^* = T^\#$.

3. **Definition and simple properties of the order-continuous operators on the L_p-spaces, $1 \leqq p \leqq \infty$.** Let (S, Σ, μ) and (S_0, Σ_0, μ_0) be two σ-finite measure spaces. If T is a linear map of $L_\infty(S, \Sigma, \mu)$ into $L_\infty(S_0, \Sigma_0, \mu_0)$, then we define

$$(3.1)\qquad \|T\|_{p,p'} = \sup_{f \in L_\infty(\mu) \cap L_p(\mu);\, \|f\|_p \neq 0} \frac{\|Tf\|_{p'}}{\|f\|_p}, \qquad 1 \leqq p, p' \leqq \infty.$$

Because $L_\infty(\mu) \cap L_p(\mu)$ is dense in $L_p(\mu)$, $1 \leqq p \leqq \infty$, it is easy to see that when (3.1) is finite, it is the norm of the unique map induced by T from $L_p(\mu)$ to $L_{p'}(\mu)$. If $p = p'$, then $\|T\|_{p,p'}$ is denoted simply $\|T\|_p$.

DEFINITION 1. *$\mathcal{O}(\mu, \mu_0)$, the set of order-continuous operators from $L_p(S, \Sigma, \mu)$ to $L_p(S_0, \Sigma_0, \mu_0)$, $1 \leqq p \leqq \infty$, is the set of all linear maps T of $L_\infty(S, \Sigma, \mu)$ into $L_\infty(S_0, \Sigma_0, \mu_0)$ such that $\|T\|_p$ is finite for $1 \leqq p \leqq \infty$. When the measure spaces are understood, they will simply be called the order-operators.*

If $T \in \mathcal{O}(\mu, \mu_0)$, then T induces another continuous linear map which will play a distinguished role in what follows. Let T_1 be the trace of T on $L_1(\mu)$. Then T_1^{**} is a continuous linear map of $\text{ba}(S, \bar{\Sigma}, \bar{\mu})$ into $\text{ba}(S, \bar{\Sigma}_0, \bar{\mu}_0)$. T_1^{**} will be called the trace of T on $\text{ba}(S, \bar{\Sigma}, \bar{\mu})$, and $\|T\|_b = \|T_1^{**}\|_b = \|T\|_1$.

In [4, p. 141 f.], N. Dunford and J. Schwartz proved a proposition concerning operators in $\mathcal{O}(\mu, \mu)$ which may be stated as follows: If $T \in \mathcal{O}(\mu, \mu)$, then there exists a uniquely determined operator $P_T \in \mathcal{O}(\mu, \mu)$ such that

(1) P_T is positive in $L_1(S, \Sigma, \mu)$;

$$(3.2)\qquad (2)\quad \|P_T\|_\infty \leqq \|T\|_\infty \quad\text{and}\quad \|P_T\|_1 \leqq \|T\|_1;$$

$$(3.3)\qquad (3)\quad |Tf| \leqq P_T|f| \qquad \text{for } f \in L_1(\mu);$$

$$(3.4)\qquad (4)\quad P_T f = \sup_{|g| \leqq f} |Tg|, \qquad 0 \leqq f \in L_\infty(\mu).$$

Inspection of the proof (given, for example, in [5, p. 672]) reveals that proposition remains valid for $T \in \mathcal{O}(\mu, \mu_0)$. Equation (3.4) implies

$$(3.5)\qquad P_T|f| = \sup_{|g| \leqq |f|} |Tg| \geqq |Tf|, \qquad f \in L_\infty(\mu).$$

Then the continuity of P_T and T and the fact that $L_\infty(\mu) \cap L_1(\mu)$ is dense in $L_p(\mu)$, $1 \leqq p \leqq \infty$, imply that

$$(3.6)\qquad P_T|f| \geqq |Tf|, \qquad f \in L_p(\mu),\ 1 \leqq p \leqq \infty.$$

This inequality implies that $\|P_T\|_p \geqq \|T\|_p$, $1 \leqq p \leqq \infty$. Hence, $\|P_T\|_1 = \|T\|_1$ and $\|P_T\|_\infty = \|T\|_\infty$. If T was positive to begin with, $P_T = T$ and

$$(3.7)\qquad P_T|f| = T|f| \geqq |Tf|, \qquad f \in L_p(\mu),\ 1 \leqq p \leqq \infty.$$

DEFINITION 2. *A sequence of complex-valued μ-measurable functions $\{f_n\}_{n=1}^\infty$ is called dominatedly order-convergent (d.o.c.) if there exists $g \in L_p(\mu)$ for some $1 \leqq p \leqq \infty$ such that $|f_n| \leqq g$, $n = 1, 2, \ldots$, and $f_n \to f$ μ a.e.*

The reader may verify that operators in $\mathcal{O}(\mu, \mu_0)$ map d.o.c. sequences into d.o.c. sequences. The proof is accomplished by showing that P_T maps dominated monotonely convergent sequences into dominated monotonely convergent sequences and then using (3.6) to obtain the desired result. The reader will, of course, recall that in this situation $f_n \to f$ μ a.e. if and only if $\lim\sup |f - f_n| = 0$.

PROPOSITION 1. *Let $T \in \mathcal{O}(\mu, \mu_0)$ and $0 \leqq f \in L_p(\mu)$ for some p such that $1 \leqq p \leqq \infty$, then*

$$P_T f = \sup_{|g| \leqq f;\, g \in L_0(\mu)} |Tg| = \sup_{|h| \leqq f;\, h \in L_p(\mu)} |Th|, \tag{3.8}$$

where $L_0(\mu)$ is the set of μ-integrable simple functions.

Proof. Clearly,

$$\sup_{|g| \leqq f;\, g \in L_0(\mu)} |Tg| \leqq \sup_{|h| \leqq f;\, h \in L_p(\mu)} |Th|. \tag{3.9}$$

On the other hand, if $|h| \leqq f$ and $h \in L_p(\mu)$ then there exists $g_n \to h$ μ a.e. such that $g_n \in L_0(\mu)$, $n = 1, 2, \ldots$, and $|g_n| \leqq |h|$, $n = 1, 2, \ldots$. $Tg_n \to Th$ μ a.e. by remarks above which implies $|Tg_n| \to |Th|$ μ a.e. Therefore,

$$\sup_n |Tg_n| \geqq |Th|. \tag{3.10}$$

So,

$$\sup_{|g| \leqq f;\, g \in L_0(\mu)} |Tg| \geqq |Th|. \tag{3.11}$$

And,

$$\sup_{|g| \leqq f;\, g \in L_0(\mu)} |Tg| \geqq \sup_{|h| \leqq f;\, h \in L_p(\mu)} |Th|. \tag{3.12}$$

This proves the right-hand equality in (3.8). (3.6) implies

$$P_T f \geqq \sup_{|h| \leqq f;\, h \in L_p(\mu)} |Th|. \tag{3.13}$$

But, if $0 \leqq f_n \uparrow f$ and $f_n \in L_\infty(\mu)$, $n = 1, 2, \ldots$, then

$$P_T f_n = \sup_{|h| \leqq f_n;\, h \in L_\infty(\mu)} |Th| = \sup_{|h| \leqq f_n;\, h \in L_p(\mu)} |Th|, \tag{3.14}$$

since $h \in L_\infty(\mu)$ and $|h| \leqq f_n \leqq f \in L_p(\mu)$ imply $h \in L_p(\mu)$. Therefore,

$$P_T f_n \leqq \sup_{|h| \leqq f;\, h \in L_p(\mu)} |Th|; \tag{3.15}$$

and we have

$$P_T f = \lim_n P_T f_n \leqq \sup_{|h| \leqq f;\, h \in L_p(\mu)} |Th|. \tag{3.16}$$

This proves the reverse inequality to (3.13) and completes the proof of the proposition. Q.E.D.

Let us observe the following fact, whose proof is left to the reader: If $S \in \mathscr{B}_p(\mu, \mu_0)$ and $T \in \mathscr{B}_{p'}(\mu, \mu_0)$, $1 \leqq p, p' < \infty$, and

$$\int S\chi_E \cdot \chi_{E_0}\mu_0(ds_0) = \int T\chi_E \cdot \chi_{E_0}\mu_0(ds_0)$$

for all $E \in \Sigma$ and $E_0 \in \Sigma_0$ such that $\mu(E) < \infty$ and $\mu_0(E_0) < \infty$, then T is the unique bounded extension of S to $L_{p'}$ and S is the unique bounded extension of T to L_p.

Defining $(T^*)_q = (T_p)^*$ for $1 < q \leqq \infty$ and $(T^*)_b = (T_\infty)^*$, let us show that these operators are the q-traces of an element in $\mathcal{O}(\mu_0, \mu)$. If $1 < q$, $q' \leqq \infty$, then $1 \leqq p$, $p' < \infty$. Therefore, if $\mu(E) < \infty$ and $\mu_0(E_0) < \infty$, then

$$\begin{aligned} \int (T^*)_q\chi_{E_0} \cdot \chi_E\mu(ds) &= \int \chi_{E_0} \cdot T_p\chi_E\mu_0(ds_0) \\ &= \int \chi_{E_0} \cdot T_{p'}\chi_E\mu_0(ds_0) \qquad (3.17) \\ &= \int (T^*)_{q'}\chi_{E_0} \cdot \chi_E\mu_0(ds_0). \end{aligned}$$

With a little additional argument and the remark above (3.17) shows that for $1 < q \leqq \infty$ the q-traces are consistent. The proof would be completed by showing that $(T^*)_b = (T_\infty)^*$ is an extension of T_q^* defined on the natural embedding κ^{-1} of the μ_0-integrable simple functions in $\mathrm{ca}(S_0, \bar{\Sigma}_0, \bar{\mu}_0)$. Therefore, it is necessary to show first that $T_q^* f \in L_1(\mu)$, where f is a μ_0-integrable simple function.

Let $B \in \Sigma$ such that $\mu(B) < \infty$. Then, as is well known,

$$\int_B |T_q^* f| \, \mu(ds) = \sup_{\mathscr{P}} \sum_{i=1}^{n} \left| \int_{B \cap E_i} T_q^* f \, \mu(ds) \right|, \qquad (3.18)$$

where $\mathscr{P}$ consists of all finite measurable partitions of B.

$$\left| \int_{B \cap E_i} T_q^* f \, \mu(ds) \right| \leqq \int P_T\chi_{B \cap E_i} \cdot |f| \mu_0(ds_0). \qquad (3.19)$$

Hence,

$$\int_B |T_q^* f| \, \mu(ds) \leqq \int P_T\chi_B \cdot |f| \, \mu_0(ds_0). \qquad (3.20)$$

Choose $B_n \uparrow S$ such that $\mu(B_n) < \infty$, $n = 1, 2, \ldots$. Then by the monotone convergence theorem

$$\begin{aligned} \int_S |T_q^* f| \, \mu(ds) &= \lim \int_{B_n} |T_q^* f| \, \mu(ds) \\ &\leqq \lim \int_{S_0} P_T\chi_{B_n} \cdot |f| \, \mu_0(ds_0) \qquad (3.21) \\ &\leqq \int_{S_0} P_T 1 \cdot |f| \mu_0(ds_0) < \infty. \end{aligned}$$

Therefore, $|T_q^* f|$ and $T_q^* f$ are in $L_1(\mu)$, where f is a μ_0-integrable simple function. A simple argument along the same lines shows that $T_b^* \kappa^{-1}(f) = \kappa^{-1}(T_q^* f)$, essentially completing the proof of the following theorem.

THEOREM 1. *$\mathcal{O}(\mu, \mu_0)$ is a linear space. Equipped with the norm*

$$\|T\|_{\mathcal{O}(\mu,\mu_0)} = \sup_{1 \leq p \leq \infty} \|T_p\|_p \tag{3.22}$$

$\mathcal{O}(\mu, \mu_0)$ is a Banach space. Further,

$$\|T\|_{\mathcal{O}(\mu,\mu_0)} = \max \{\|T_1\|_1, \|T_\infty\|_\infty\}. \tag{3.23}$$

There is a linear isometric isomorphism of $\mathcal{O}(\mu, \mu_0)$ onto $\mathcal{O}(\mu_0, \mu)$, denoted "$*$", having the following properties:

(3.24) (1) $(T^*)_q = (T_p)^*$, $1 \leq p < \infty$, where $1/p + 1/q = 1$;

(3.25) (2) $(T^*)_b = (T_\infty)^*$;

(3.26) (3) $T^{**} = T$;

(4) "$*$" *takes Banach positive operators into Banach positive operators.*

Proof. (3.22) is immediate. (3.23) follows from the Riesz convexity theorem. (3.24) and (3.25) were demonstrated in the remarks preceding the theorem. The completeness of $\mathcal{O}(\mu, \mu_0)$ is a consequence of the completeness of the space $\mathcal{B}_p(\mu, \mu_0)$, $1 \leq p \leq \infty$. (3.26) is a simple consequence of the definitions and shows that "$*$" is into. (4) is a consequence of very simple duality arguments.

It should be noted that T_1^* is created by restricting $(T_\infty)^*$ to $\mathrm{ca}(S_0, \bar{\Sigma}_0, \bar{\mu}_0)$. Therefore, $(T^*)_b = (T_1^*)^{**}$ is the uniquely determined extension of T_1^* in $(L_1(S_0, \Sigma_0, \mu_0))^{**}$. But this implies that $(T^*)_b = (T_\infty)^*$. The isometric isomorphism part of the preceding theorem now follows from $\|T_p\|_p = \|T_q^*\|_q$, $1/p + 1/q = 1$. Q.E.D.

COROLLARY 1. *If $T \in \mathcal{O}(\mu, \mu_0)$,*

$$P_{T^*} = (P_T)^*. \tag{3.27}$$

Proof. Let $g \in L_\infty(\mu) \cap L_1(\mu)$ and $f \in L_\infty(\mu_0) \cap L_1(\mu_0)$. Then,

$$\begin{aligned} \left|\int Tg \cdot f \mu_0(ds_0)\right| &= \left|\int g \cdot T^* f \mu(ds)\right| \\ &\leq \int |g| \cdot P_{T^*}|f| \mu(ds) \\ &\leq \int (P_{T^*})^* |g| \cdot |f| \mu_0(ds_0). \end{aligned} \tag{3.28}$$

Using the fact that the total variation of $\kappa^{-1}(Tg) = \kappa^{-1}(|Tg|)$ and the argument leading to equation (3.20), we have

$$|Tg| \leq (P_{T^*})^* |g|, \qquad g \in L_\infty(\mu) \cap L_1(\mu). \tag{3.29}$$

Hence,

$$\begin{aligned} P_T|g| &= \sup_{h\in L_\infty(\mu);\,|h|\leqq|g|} |Th| \\ &\leqq \sup_{h\in L_\infty(\mu);\,|h|\leqq|g|} (P_{T^*})^*|h| \\ &\leqq (P_{T^*})^*|g|, \qquad g\in L_\infty(\mu)\cap L_1(\mu). \end{aligned} \tag{3.30}$$

This implies

$$(P_T)^* \leqq P_{T^*}. \tag{3.31}$$

Substituting T^* for T in (3.31),

$$(P_{T^*})^* \leqq P_T, \tag{3.32}$$

implying

$$P_{T^*} \leqq (P_T)^*. \tag{3.33}$$

Q.E.D.

COROLLARY 2. *$\mathcal{O}(\mu,\mu)$ with the conjugate linear adjoint operation $T\to T^\#$ is a Banach star algebra whose norm satisfies the symmetry condition*

$$\|T\|_{\mathcal{O}(\mu,\mu)} = \|T^\#\|_{\mathcal{O}(\mu,\mu_0)}. \tag{3.34}$$

Proof. Immediately, the corollary follows from the fact that if $S, T\in \mathscr{B}_p(\mu,\mu)$,

$$\|ST\|_p \leqq \|S\|_p\|T\|_p. \tag{3.35}$$

Q.E.D.

Let us observe that Proposition 1 implies that for $0\leqq f\in L_p(\mu)$

$$\begin{aligned} P_{T^\#}f &= \sup_{|g|\leqq f;\, g\in L_p(\mu)} |(j\circ T^*\circ j)g| \\ &= \sup_{|g|\leqq f;\, g\in L_p(\mu)} |T^*g| = P_{T^*}f. \end{aligned} \tag{3.36}$$

But,

$$P_T^\# = P_T^*. \tag{3.37}$$

Hence, Corollary 1 implies

$$P_{T^\#} = P_T^\#. \tag{3.38}$$

Thus, the positive of an element $T\in\mathcal{O}(\mu,\mu)$ such that $T=T^\#$ has the property $P_T^\#=P_T$.

PROPOSITION 2. *If E is an idempotent in $\mathcal{O}(\mu,\mu)$ such that $E^\#=E$, and $\|E\|_{\mathcal{O}(\mu,\mu)}\leqq 1$, then P_E is an idempotent, $P_E^\#=P_E$ and $\|P_E\|_{\mathcal{O}(\mu,\mu)}\leqq 1$.*

Proof. It is easy to show from definition that $P_{T_1T_2}\leqq P_{T_1}P_{T_2}$ for any T_1, $T_2\in\mathcal{O}(\mu,\mu)$. Hence,

$$P_E = P_{EE} \leqq P_EP_E. \tag{3.39}$$

Because $\|P_E\|_{\mathcal{O}(\mu,\mu)}=\max\{\|E\|_1, \|E\|_\infty\}\leqq 1$, (3.39) implies $P_E=P_E^2$. By the remark above,

$$P_E^{\#} = P_{E^{\#}} = P_E. \tag{3.40}$$

Q.E.D.

4. **Representation of $\mathcal{O}(\mu,\mu)$ in $\mathcal{B}_2(\mu,\mu)$.** Let τ^p be the natural mapping of $\mathcal{O}(\mu,\mu_0)$ into $\mathcal{B}_p(\mu,\mu_0)$ determined by $\tau^p(T)=T_p$, the p-trace of T. τ_0^p is the analogous mapping which takes $\mathcal{O}(\mu_0,\mu)$ into $\mathcal{B}_p(\mu_0,\mu)$. τ^p and τ_0^p are linear, continuous, norm-reducing mappings. τ^p and τ_0^p are one-to-one. For if $S, T\in\mathcal{O}(\mu,\mu_0)$ and

$$\tau^p(T) = \tau^p(S), \tag{4.1}$$

then

$$Tf = Sf, \qquad f\in L_\infty(\mu)\cap L_1(\mu). \tag{4.2}$$

By a now-familiar argument, (4.2) implies $S=T$. Further, τ^p and τ_0^p map (Banach) positive operators into (Banach) positive operators;

$$(\tau^p(T))^* = \tau^p(T^*); \tag{4.3}$$

and

$$(\tau^p(T))^{\#} = \tau_0^p(T^{\#}). \tag{4.4}$$

In the case $(S,\Sigma,\mu)=(S_0,\Sigma_0,\mu_0)$ and $p=2$, the remarks above imply that τ^2 is a faithful star-representation of $\mathcal{O}(\mu,\mu)$ on $\mathcal{B}_2(\mu,\mu)$. Using the terminology of Rickart [10], $\mathcal{O}(\mu,\mu)$ with the conjugate-linear adjoint operation is an A^*-algebra because it possesses the auxiliary norm $\|\cdot\|_2$ satisfying the B^*-condition. Thus, $\mathcal{O}(\mu,\mu)$ has radical equal to $\{0\}$; and $\mathcal{O}(\mu,\mu)$ as well as any of its star-subalgebras is semisimple. Because of their importance, these remarks are summarized in the following theorem.

THEOREM 2. *$\mathcal{O}(\mu,\mu)$ equipped with the conjugate-linear adjoint operation is an A^*-algebra. The representation on Hilbert space given by τ^2 is faithful.*

Proof. These statements follow from the remarks preceding the theorem. Q.E.D.

If (S,Σ,μ) is a σ-finite measure space on a finite set of points or is isomorphic to such a space, then $\mathcal{O}(\mu,\mu)=\mathcal{B}_1(\mu,\mu)=\mathcal{B}_\infty(\mu,\mu)$ as point sets.

For a general σ-finite measure space (S,Σ,μ), define

$$T_f g = f\cdot g, \tag{4.5}$$

where $f\in L_\infty(\mu)$ and g is a μ-integrable simple function. It can be proved that $T_f\in\mathcal{O}(\mu,\mu)$ and $\|T_f\|_p=\|f\|_\infty$. Let $\mathcal{M}=\{T_f \mid f\in L_\infty(\mu)\}$. Segal [12] has shown that $\mathcal{M}$ acting on $L_2(\mu)$ is a maximal abelian subalgebra of $\mathcal{B}_2(\mu,\mu)$. Thus, any element of $\mathcal{B}_2(\mu,\mu)$ which commutes with $\tau^2(\mathcal{O}(\mu,\mu))$ commutes with $\mathcal{M}$, and so is a member of $\mathcal{M}$ by maximality.

It is possible to define an operator on the σ-finite measure space (S,Σ,μ) which is the analog of conditional expectation on a sub-σ-algebra. If Σ_1 is a sub-σ-algebra of Σ containing μ-null sets and such that $\mu|_{\Sigma_1}$, the restriction of μ to Σ_1, is σ-finite, then define

$$E^{\Sigma_1}f = \frac{d}{d\mu|_{\Sigma_1}}(\kappa^{-1}(f)|_{\Sigma_1}), \tag{4.6}$$

f a μ-integrable simple function. It is possible to show that $E^{\Sigma_1} \in \mathcal{O}(\mu, \mu)$ and that E^{Σ_1} has the properties of the conditional expectation operator. However, for the purposes of the following theorem it suffices to show the existence of this operator in a very simple case.

Corresponding to each partition $\Sigma_1 = \{C_i\}_{i=1}^{\infty}$ of S by sets of finite, positive μ-measure there is an operator $E^{\Sigma_1} \in \mathcal{O}(\mu, \mu)$ defined by

$$E^{\Sigma_1}f(s) = \sum_{i=1}^{\infty} \frac{\int_{C_i} f(s')\mu(ds')}{\mu(C)} \chi_{C_i}(s). \tag{4.6}$$

Clearly, E^{Σ_1} induces a contraction on $L_\infty(S, \Sigma, \mu)$. Moreover, if $f \in L_p(S, \Sigma, \mu)$,

$$\|E^{\Sigma_1}f\|_p = \left\{\sum_{i=1}^{\infty} |\alpha_i|^p \mu(C_i)\right\}^{1/p}, \tag{4.7}$$

where

$$|\alpha_i|^p = \left|\int_{C_i} f(s') \frac{\mu(ds')}{\mu(C_i)}\right|^p \leqq \int_{C_i} |f(s')|^p \frac{\mu(ds')}{\mu(C_i)} \tag{4.8}$$

by a convexity inequality valid for $1 \leqq p < \infty$ (see [7, p. 159]). But then,

$$\begin{aligned} \|E^{\Sigma_1}f\|_p &\leqq \left\{\sum_{i=1}^{\infty} \int_{C_i} |f(s')|^p \frac{\mu(ds')}{\mu(C_i)} \mu(C_i)\right\}^{1/p} \\ &\leqq \left\{\sum_{i=1}^{\infty} \int_{C_i} |f(s')|^p \mu(ds')\right\}^{1/p} \\ &\leqq \|f\|_p. \end{aligned} \tag{4.9}$$

E^{Σ_1} induces a contraction in all the L_p-spaces, $1 \leqq p < \infty$. Since E^{Σ_1} is obviously a contraction in $L_\infty(\mu)$, $E^{\Sigma_1} \in \mathcal{O}(\mu, \mu)$.

If (S, Σ, μ) is a finite measure space, there exist contraction operators of the foregoing type constructed on finite as well as countable partitions. In particular, there is an operator associated with the partition consisting of S alone. This operator is nothing more than integration with respect to the measure, properly normalized. Let $\mathcal{S}$ be the set of partition operators, where finite partitions are allowed if the measure space is finite. The proof of the following simple fact is left to the reader: If $E^{\Sigma_1}f = f$ for all $E^{\Sigma_1} \in \mathcal{S}$, then $f = \alpha 1$, α a complex scalar.

THEOREM 3. *If $T \in \mathcal{B}_2(\mu, \mu)$ and T commutes with $\tau^2(\mathcal{M} \cup \mathcal{S})$, then*

$$T = \alpha I, \tag{4.10}$$

where α is a complex scalar.

Proof. T commutes with $\tau^2(\mathscr{M})$ implies $T=T_f$ for some $f\in L_\infty(\mu)$. But then T is the trace of an element in $\mathcal{O}(\mu,\mu)$ and $f=T1$.

T commutes with $\tau^2(\mathscr{S})$ implies $E^{\Sigma_1}Tg=TE^{\Sigma_1}g$ for all $g\in L_\infty(\mu)\cap L_1(\mu)$ and $E^{\Sigma_1}\in\mathscr{S}$. The same relation holds for any $h\in L_p(\mu)$, $1\leqq p\leqq\infty$, since E^{Σ_1} and T are elements of $\mathcal{O}(\mu,\mu)$. On the other hand, for all $E^{\Sigma_1}\in\mathscr{S}$, $E^{\Sigma_1}1=1$ and

$$f = T1 = TE^{\Sigma_1}1 = E^{\Sigma_1}T1 = E^{\Sigma_1}f. \tag{4.11}$$

By the remark preceding the theorem, $f=\alpha 1$, proving the assertion. Q.E.D.

COROLLARY 1. *The representation* $\tau^2\colon \mathcal{O}(\mu,\mu)\to\mathscr{B}_2(\mu,\mu)$ *is irreducible, and* $\tau^2(\mathcal{O}(\mu,\mu))$ *is weakly dense in* $\mathscr{B}_2(\mu,\mu)$.

Proof. The first assertion follows from the theorem noting that $(\mathscr{M}\cup\mathscr{S})\subset\mathcal{O}(\mu,\mu)$. The second assertion is a consequence of the von Neumann density theorem (see, for example, Corollary 4 [8, p. 448]).

5. **Various topologies.** The continuous mappings $\tau^p\colon \mathcal{O}(\mu,\mu_0)\to\mathscr{B}_p(\mu,\mu_0)$, $1\leqq p\leqq\infty$, endow $\mathcal{O}(\mu,\mu_0)$ with a bewildering number of topologies. The purpose of this section is to show that some of these topologies are equivalent on $\mathcal{O}^1$, the closed unit sphere of $\mathcal{O}(\mu,\mu_0)$. Some useful convergence theorems are obtained in the process.

Let $\mathscr{F}$ be the topology induced on $\mathcal{O}(\mu,\mu_0)$ by the linear functionals

$$\varphi(T) = \int Tf\cdot g\mu_0(ds_0), \tag{5.1}$$

where $f\in L_\infty(\mu)\cap L_1(\mu)$ and $g\in L_\infty(\mu_0)\cap L_1(\mu_0)$. For $1<p<\infty$, $\mathscr{T}_p$ is the topology induced in $\mathcal{O}(\mu,\mu_0)$ by the linear functionals

$$\psi(T) = \int Th\cdot k\mu_0(ds_0), \tag{5.2}$$

where $h\in L_p(\mu)$ and $k\in L_q(\mu)$, $1/p+1/q=1$, $\mathscr{T}_p$ is nothing more than the weak operator topology of $\mathscr{B}_p(\mu,\mu_0)$ pulled back into $\mathcal{O}(\mu,\mu_0)$ by the mapping τ^p. $\mathscr{F}\subset\mathscr{T}_p$, $1<p<\infty$, and all of these topologies are Hausdorff. The closed unit sphere of $\mathscr{B}_p(\mu,\mu_0)$ is known to be compact in the weak topology, $1<p<\infty$ [5, p. 512]. Since τ^p is a contraction, $\tau^p(\mathcal{O}^1)$ would be a compact convex set if $\tau^p(\mathcal{O}^1)$ is weakly closed. But the identity mapping of $\mathcal{O}^1$ with the $\mathscr{T}_p$ topology onto $\mathcal{O}^1$ with the $\mathscr{F}$ topology is a continuous one-to-one mapping of a compact space onto a Hausdorff space. Hence, it would be a homeomorphism and all the topologies $\mathscr{F}$ and $\mathscr{T}_p$, $1<p<\infty$, would be equivalent on $\mathcal{O}^1$.

THEOREM 4. $\mathcal{O}^1$ *is a compact convex subset of* $\mathcal{O}(\mu,\mu_0)$ *equipped with any one of the topologies* $\mathscr{F}$ *or* $\mathscr{T}_p$, $1<p<\infty$. *The restrictions of these topologies to* $\mathcal{O}^1$ *are equivalent. This topology is metric if* (S,Σ,μ) *and* (S_0,Σ_0,μ_0) *are essentially countable generated.*

Proof. By the preceding remarks, it suffices to show that $\tau^p(\mathcal{O}^1)$ is closed in the weak operator topology of $\mathscr{B}_p(\mu, \mu_0)$ for each p, $1<p<\infty$.

Suppose the net $\{\tau^p(T_\alpha)\}_{\alpha\in A}$ converges to $T\in\mathscr{B}_p(\mu, \mu_0)$ in the weak operator topology of $\mathscr{B}_p(\mu, \mu_0)$. If $1<p'<\infty$, every subnet of $\{\tau^{p'}(T_\alpha)\}_{\alpha\in A}$ contains a subnet convergent in the weak operator topology of $\mathscr{B}_{p'}(\mu, \mu_0)$. Let the limit of any such convergent subnet be denoted $S\in\mathscr{B}_{p'}(\mu, \mu_0)$. Then,

$$\int Tf\cdot g\mu_0(ds_0) = \int Sf\cdot g\mu_0(ds_0), \tag{5.3}$$

for all $f\in L_\infty(\mu)\cap L_1(\mu)$ and all $g\in L_\infty(\mu_0)\cap L_1(\mu_0)$. By an application of an argument based on the Radon-Nikodym theorem,

$$Tf = Sf, \qquad f\in L_\infty(\mu)\cap L_1(\mu). \tag{5.4}$$

Thus, T has a unique bounded extension in $\mathscr{B}_{p'}(\mu, \mu_0)$ which is equal to S. Let this extension of T to $L_{p'}(\mu)$ also be called T. Then the preceding argument shows that T is the only accumulation point of $\{\tau^{p'}(T_\alpha)\}_{\alpha\in A}$. Hence, $\{\tau^{p'}(T_\alpha)\}_{\alpha\in A}$ converges to T in the weak operator topology of $\mathscr{B}_{p'}(\mu, \mu_0)$, $1<p'<\infty$. The adjoint operation is weakly continuous. Therefore, $\{\tau_0^q(T_\alpha^*)\}_{\alpha\in A}$ converges to T^* in the weak operator topology of $\mathscr{B}_q(\mu, \mu)$, $1<q<\infty$. From this it follows that $\|T\|_p\leqq 1$ and $\|T^*\|_q\leqq 1$, $1<p, q<\infty$. If $\chi_{E_m}\uparrow\chi_{S_0}$ is such that $\mu_0(E_m)<\infty$, $m=1, 2, \ldots$, then for $f\in L_\infty(\mu)\cap L_1(\mu)$ and $g\in L_\infty(\mu_0)$

$$\lim_{\alpha\in A}\int_{E_m} T_\alpha f\cdot g\mu_0(ds_0) = \int_{E_m} Tf\cdot g\mu_0(ds_0). \tag{5.5}$$

Thus, $\{\chi_{E_m}T_\alpha f\}_{\alpha\in A}$ converges weakly in $L_1(\mu_0)$ to the function $\chi_{E_m}Tf$. And

$$\|\chi_{E_m}Tf\|_1 \leqq \sup_{\alpha\in A}\|\chi_{E_m}T_\alpha f\|_1 \leqq 1\cdot\|f\|_1, \tag{5.6}$$

for $m=1, 2, \ldots$. But,

$$\int |Tf|\,\mu_0(ds_0) = \lim_m \int \chi_{E_m}\,|Tf|\,\mu_0(ds_0) \leqq 1\cdot\|f\|_1. \tag{5.7}$$

This implies

$$\|Tf\|_1 \leqq 1\cdot\|f\|_1 \tag{5.8}$$

on a set dense in $L_1(\mu)$. Therefore, $\|T\|_1\leqq 1$ and T has an extension to $L_1(\mu)$. An analogous result holds for T^*.

Denote T^* in $L_1(\mu_0)$ by $(T^*)_1$; then $(T^*)_1^*$ is a bounded linear operator in $\mathscr{B}_\infty(\mu, \mu_0)$ of norm less than or equal to one. However,

$$\begin{aligned}\int (T^*)_1^* f\cdot g\,\mu_0(ds_0) &= \int f\cdot(T^*)_1 g\mu(ds)\\ &= \int f\cdot T^*g\mu(ds) = \int Tf\cdot g\mu_0(ds_0)\end{aligned} \tag{5.9}$$

for $f \in L_\infty(\mu) \cap L_1(\mu)$ and $g \in L_\infty(\mu_0) \cap L_1(\mu_0)$. Omitting an argument used several times above, T has a (unique) bounded extension to $L_\infty(\mu)$ having norm less than or equal to one. Therefore, $T \in \mathcal{O}^1$ and $\tau^p(\mathcal{O}^1)$ is closed in the weak operator topology of $\mathscr{B}_p(\mu, \mu_0)$.

If (S, Σ, μ) and (S_0, Σ_0, μ_0) are essentially countably generated, then $L_p(\mu)$ and $L_q(\mu_0)$ are separable for $1 < p < \infty$ and $1/p + 1/q = 1$. Let $\{f_n\}_{n=1}^\infty$ and $\{g_m\}_{m=1}^\infty$ be dense sequences in $L_p(\mu)$ and $L_q(\mu_0)$ respectively. The series

$$\rho(S, T) = \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{1}{2^{n+m}} \cdot \frac{|\psi_{n,m}(S-T)|}{1+|\psi_{n,m}(S-T)|} \tag{5.10}$$

is a metric on $\mathcal{O}^1$, where

$$\psi_{n,m}(S-T) = \int (S-T)f_n \cdot g_m \mu_0(ds_0). \tag{5.11}$$

Since the identity map of $\mathcal{O}^1$ with the $\mathscr{F}$ topology is one-to-one and continuous onto $\mathcal{O}^1$ with the ρ topology which is Hausdorff, the two topologies are equivalent. Q.E.D.

The collection of functionals which induce the $\mathscr{F}$ topology on $\mathcal{O}^1$ can be greatly "thinned out" without changing the topology. Since $\mathcal{O}^1$ with the $\mathscr{F}$ topology is compact, all that is necessary for equivalence is to show that "thinned out" topology is Hausdorff.

Let Φ and Φ_0 be collections of sets of finite measure such that span $\{\chi_E \mid E \in \Phi\}$ and span $\{\chi_{E_0} \mid E_0 \in \Phi_0\}$ are dense in $L_p(\mu)$ and $L_q(\mu_0)$, respectively, where $1/p + 1/q = 1$ and $1 < p < \infty$. Consider the topology induced by the functionals

$$\eta(T) = \int T\chi_E \cdot \chi_{E_0} \mu_0(ds_0), \tag{5.12}$$

$E \in \Phi$ and $E_0 \in \Phi_0$. This is the same as the topology induced by the functionals

$$\varphi(T) = \int Tf \cdot g \mu_0(ds_0), \tag{5.13}$$

where $f \in$ span $\{\chi_E \mid E \in \Phi\}$ and $g \in$ span $\{\chi_{E_0} \mid E_0 \in \Phi_0\}$. If $S, T \in \mathcal{O}^1$ and S and T cannot be separated by the functionals φ, then

$$\begin{aligned} |\langle Tf, g\rangle - \langle Sf, g\rangle| \leq\ & |\langle Tf, g\rangle - \langle Tf, h_n\rangle| \\ & + |\langle Tf, h_n\rangle - \langle Sf, h_n\rangle| \\ & + |\langle Sf, h_n\rangle - \langle Sf, g\rangle|, \end{aligned} \tag{5.14}$$

where $f \in$ span $\{\chi_E \mid E \in \Phi\}$, $g \in L_\infty(\mu_0) \cap L_1(\mu_0)$, and $\{h_n\}_{n=1}^\infty \subset$ span $\{\chi_{E_0} \mid E_0 \in \Phi_0\}$ with $\lim_n \|g - h_n\|_q = 0$. The middle term on the right-hand side is zero by hypothesis. The first and third terms can be made arbitrarily small for sufficiently large n by the inequalities

$$|\langle Tf, g\rangle - \langle Tf, h_n\rangle| \leq \|f\|_p \|g - h_n\|_q \tag{5.15}$$

and

$$|\langle Sf, h_n\rangle - \langle Sf, g\rangle| \leqq \|f\|_p \|h_n - g\|_q. \tag{5.16}$$

Therefore,

$$\langle Tf, g\rangle = \langle Sf, g\rangle. \tag{5.17}$$

But $L_\infty(\mu_0) \cap L_1(\mu_0)$ includes the indicator functions of all sets of finite measure. By application of an argument based on the Radon-Nikodym theorem, $Tf = Sf$ for f contained in a set dense in $L_p(\mu)$. This implies $S = T$ in $\mathcal{O}^1$, and the topology induced by Φ and Φ_0 is equivalent to the $\mathcal{F}$ topology on $\mathcal{O}^1$.

PROPOSITION 3. *If $\{T_\alpha\}_{\alpha \in A}$ is a net in $\mathcal{O}^1$, then $\{T_\alpha\}_{\alpha\in A}$ converges to some $T \in \mathcal{O}^1$ in the $\mathcal{F}$ topology (and, hence, in the $\mathcal{T}_p$ topology, $1 < p < \infty$) if and only if*

$$\lim_{\alpha \in A} \int \chi_{E_0} \cdot T_\alpha \chi_E \mu_0(ds_0) \tag{5.18}$$

exists and is finite for all $E \in \Phi$ and $E_0 \in \Phi_0$. Under condition (5.18), *$T_\alpha^* \to T^*$ in the $\mathcal{F}$ topology of the closed unit sphere of $\mathcal{O}(\mu_0, \mu)$.*

Proof. The functionals η of (5.12) induce a uniformity on $\mathcal{O}(\mu, \mu_0)$ and a relative uniformity on $\mathcal{O}^1$. With respect to this relative uniform topology $\mathcal{O}^1$ is a compact space by remarks preceding the proposition. Therefore, $\mathcal{O}^1$ in this relative uniform topology is complete. (5.18) asserts that $\{T_\alpha\}_{\alpha \in A}$ is a Cauchy net in the relative uniform topology. Hence, $T_\alpha \to T \in \mathcal{O}^1$ in this topology. But this relative uniform topology is equivalent to the $\mathcal{F}$ topology on $\mathcal{O}^1$. Therefore, $T_\alpha \to T$ in the $\mathcal{F}$ topology. The last assertion follows from an application of the result above to the net $\{T_\alpha^*\}_{\alpha \in A}$, noting that

$$\int T_\alpha^* \chi_{E_0} \cdot \chi_E \mu(ds) = \int \chi_{E_0} \cdot T_\alpha \chi_E \mu_0(ds_0). \tag{5.19}$$ Q.E.D.

$\mathcal{T}_1$ denotes the weak operator topology in $\mathcal{B}_1(\mu, \mu_0)$ pulled back into $\mathcal{O}(\mu, \mu_0)$ by τ^1. It is induced by the functionals

$$\psi(T) = \int Tf \cdot g \mu_0(ds_0), \tag{5.20}$$

$f \in L_1(\mu)$ and $g \in L_\infty(\mu_0)$. Let Φ' be a collection of sets of finite measure in Σ such that span $\{\chi_E \mid E \in \Phi'\}$ is dense in $L_1(\mu)$. Then span $\{\chi_E \mid E \in \Phi'\}$ is dense in $L_p(\mu)$, $1 < p < \infty$. For if $f \in L_\infty(\mu) \cap L_1(\mu)$, then there exists $\{f_n\}_{n=1}^\infty \subset$ span $\{\chi_E \mid E \in \Phi'\}$ such that $\|f_n\|_\infty \leqq \|f\|_\infty$, $n = 1, 2, \ldots$, and $\lim_n \|f - f_n\|_1 = 0$. By 6 Theorem [5, p. 122] $f_n \to f$ in μ-measure. Further,

$$\lim_{\mu(E)\to 0} \int_E |f_n|^p \mu(ds) \leqq \lim_{\mu(E)\to\infty} \left(\|f\|_\infty^{p-1} \int_E |f_n| \mu(ds) \right) \tag{5.21}$$

and

$$\int_{S-E} |f_n|^p \mu(ds) \leqq \|f\|_\infty^{p-1} \int_{S-E} |f_n| \mu(ds). \tag{5.22}$$

Because $\{f_n\}_{n=1}^{\infty}$ converges in $L_1(\mu)$, the limit in (5.21) is zero. For the same reason, by proper choice of E, the right-hand side of (5.22) can be made arbitrarily small uniformly in n. The theorem cited then implies $\lim_n \|f-f_n\|_p=0$. Hence, span $\{\chi_E \mid E\in\Phi'\}$ is dense in $L_p(\mu)$.

Since span $\{\chi_{E_0}\in\Phi'\}$ and span $\{\chi_{E_0} \mid E_0\in\Sigma_0\}$ are dense in $L_1(\mu)$ and $L_\infty(\mu_0)$ respectively, a simple argument similar to (5.14) applied twice shows that $\mathcal{O}^1$ with the $\mathcal{T}_1$ topology is equivalent to $\mathcal{O}^1$ with the topology induced by the functionals

$$\varphi(T) = \int \chi_{E_0}\cdot T\chi_E\mu_0(ds_0), \tag{5.23}$$

where $E\in\Phi'$ and $E_0\in\Sigma_0$. Let us note in preparation for the following proposition that $\mathcal{F}\subset\mathcal{T}_1$.

Proposition 4. *$\mathcal{O}^1$ with the $\mathcal{T}_1$ topology is convex, closed, and sequentially complete. It is not, in general, compact.*

A sequence $\{T_n\}_{n=1}^{\infty}\subset\mathcal{O}^1$ converges to some $T\in\mathcal{O}^1$ in the $\mathcal{T}_1$ topology if and only if

$$\lim_n \int \chi_{E_0}\cdot T_n\chi_E\mu_0(ds_0) \tag{5.24}$$

exists and is finite for $E\in\Phi'$ and $E_0\in\Sigma_0$. If (S_0, Σ_0, μ_0) is a finite measure space, $\mathcal{O}^1$ with the $\mathcal{T}_1$ topology is compact and equivalent to $\mathcal{O}^1$ with the $\mathcal{F}$ topology.

Proof. $\mathcal{O}^1$ is convex and if the net $\{T_\alpha\}_{\alpha\in A}\subset\mathcal{O}^1$ converges to some T contained in the $\mathcal{T}_1$ closure of $\mathcal{O}^1$ in $\mathcal{O}(\mu,\mu_0)$, then $T_\alpha\to T$ in the $\mathcal{F}$ topology. But this implies $T\in\mathcal{O}^1$, and that $\mathcal{O}^1$ is closed in the $\mathcal{T}_1$ topology.

If $f\in L_\infty(\mu)\cap L_1(\mu)$ and $\{T_n\}_{n=1}^{\infty}$ is a $\mathcal{T}_1$ Cauchy sequence in $\mathcal{O}^1$, then $\{T_nf\}_{n=1}^{\infty}$ is a weak Cauchy sequence in $L_1(\mu_0)$ with some weak limit $g_1\in L_1(\mu_0)$, since $L_1(\mu_0)$ is weakly sequentially complete. On the other hand, $\{T_n\}_{n=1}^{\infty}$ converges in the $\mathcal{F}$ topology to some $T\in\mathcal{O}^1$. Therefore for each $E_0\in\Sigma_0$ such that $\mu_0(E_0)<\infty$,

$$\int_{E_0} g_1\mu_0(ds_0) = \int_{E_0} Tf\mu_0(ds_0). \tag{5.25}$$

If $Tf\neq g_1$ on some set of nonzero measure, then there exists a set $F_0\in\Sigma_0$ such that $\mu_0(F_0)<\infty$ and $Tf\neq g_1$ on F_0. Then for all $E_0\in\Sigma_0$

$$\int_{F_0\cap E_0} g_1\mu_0(ds_0) = \int_{F_0\cap E_0} Tf\mu_0(ds_0). \tag{5.26}$$

By an argument based on the Radon-Nikodym theorem

$$\chi_{F_0}g_1 = \chi_{F_0}Tf, \tag{5.27}$$

and this is a contradiction unless $\mu_0(F_0)=0$ implying that $Tf=g_1$. Hence,

$$\lim_n \langle T_nf, g\rangle = \langle Tf, g\rangle \tag{5.28}$$

for $f \in L_\infty(\mu) \cap L_1(\mu)$ and $g \in L_\infty(\mu_0)$. Again by the fact that $L_\infty(\mu) \cap L_1(\mu)$ is dense in $L_1(\mu)$, it follows that $T_n \to T$ in the $\mathscr{T}_1$ topology, proving that $\mathcal{O}^1$ with the $\mathscr{T}_1$ topology is sequentially complete.

To prove that $\mathcal{O}^1$ is not, in general, compact in the $\mathscr{T}_1$ topology, consider the real line with Lebesgue measure μ. Define

$$T_n f(s) = f(s-n), \qquad n = 1, 2, \ldots. \tag{5.29}$$

Because Lebesgue measure is translation invariant, T_n preserves norm in all the L_p spaces and is a member of $\mathcal{O}^1 \subset \mathcal{O}(\mu, \mu)$. The sequence $\{T_n\}_{n=1}^\infty$ is $\mathscr{F}$ convergent to 0. If any subnet $\{T_{n_\alpha}\}_{\alpha \in A}$ converged $\mathscr{T}_1$ to some $T \in \mathcal{O}^1$, then $T=0$. But if χ_E is such that $0 < \mu(E) < \infty$, then

$$\langle T_{n_\alpha}\chi_E, 1\rangle = \mu(E). \tag{5.30}$$

Hence, no subnet can converge $\mathscr{T}_1$ to 0. $\mathcal{O}^1$ with the $\mathscr{T}_1$ topology is certainly not compact.

The convergence criterion (5.24) is a direct consequence of remarks preceding the proposition and the proof of sequential completeness given above.

Finally, suppose (S_0, Σ_0, μ_0) is a finite measure space. $L_\infty(\mu_0) = L_\infty(\mu_0) \cap L_1(\mu_0)$ and $\mathscr{F}$ convergence of a net implies $\mathscr{T}_1$ convergence. Therefore the identity map of $\mathcal{O}^1$ with the $\mathscr{F}$ topology onto $\mathcal{O}^1$ with the $\mathscr{T}_1$ topology is continuous. This implies equivalence. Q.E.D.

$\mathcal{O}(\mu, \mu_0)$ inherits two weak topologies from L_∞. The first, denoted by $\mathscr{T}_1^*$, is induced by the functionals

$$\psi(T) = \int Tf \cdot g\mu_0(ds_0), \tag{5.31}$$

where $f \in L_\infty(\mu)$ and $g \in L_1(\mu_0)$. The adjoint operation "*" is a linear homeomorphism of $\mathcal{O}^1$ with the $\mathscr{T}_1^*$ topology to $\mathcal{O}^1(\mu_0, \mu)$, the closed unit sphere in $\mathcal{O}(\mu_0, \mu)$, equipped with the $\mathscr{T}_1$ topology. Thus, Proposition 4 applied to $\mathcal{O}^1(\mu_0, \mu)$ gives properties of $\mathcal{O}^1$ in the $\mathscr{T}_1^*$ topology.

The second topology $\mathcal{O}(\mu, \mu_0)$ inherits from L_∞, denoted $\mathscr{T}_\infty$, is the weak operator topology of $\mathscr{B}_\infty(\mu, \mu_0)$ pulled back by τ_∞. Since κ^{-1} injects $L_1(\mu_0)$ into $\mathrm{ba}(S_0, \bar{\Sigma}_0, \bar{\mu}_0)$, $\mathscr{F} \subset \mathscr{T}_1^* \subset \mathscr{T}_\infty$.

Proposition 5. *$\mathcal{O}^1$ with the $\mathscr{T}_\infty$ topology is convex and closed. A sequence $\{T_n\}_{n=1}^\infty \subset \mathcal{O}^1$ converges to $T \in \mathcal{O}^1$ with respect to the $\mathscr{T}_\infty$ topology if and only if*

$$\lim_n \langle T_n f, \lambda\rangle = \langle Tf, \lambda\rangle \tag{5.32}$$

for $f \in L_\infty(\mu)$ and $\lambda \in \mathrm{ba}(S_0, \bar{\Sigma}_0, \bar{\mu}_0)$ such that λ assumes only the values zero and one.

Proof. The proof that $\mathcal{O}^1$ is convex and closed is similar to the proof of this fact given in Proposition 4. The second assertion follows from noting that $\{T_n f\}_{n=1}^{\infty}$ is bounded in $L_\infty(\mu_0)$ and that the extreme points of the unit sphere of $L_\infty^*(\mu_0)$ are of the form $\alpha\lambda$, where α is a complex scalar of modulus one. Then, the second assertion follows from a necessary and sufficient condition for weak convergence of a bounded sequence in a normed linear space given by Rainwater [9]. Q.E.D.

Let $\mathscr{U}_p$ be the topology obtained in $\mathcal{O}(\mu, \mu_0)$ by pulling back the strong operator topology in $\mathscr{B}_p(\mu, \mu_0)$ by means of the isomorphism τ^p.

PROPOSITION 6. *$\mathcal{O}^1$ is a closed, convex subset $\mathcal{O}(\mu, \mu_0)$ with the $\mathscr{U}_p$ topology, $1 \leqq p \leqq \infty$. $\mathcal{O}^1$ is complete in the $\mathscr{U}_p$ topology, $1 \leqq p < \infty$.*

Proof. $L_p(\mu)$ and $L_p(\mu_0)$ are Banach spaces for $1 \leqq p \leqq \infty$. This implies that they are locally convex, Hausdorff, and barrelled. By Corollary 2, Theorem 4, §3, Chapter III [1], $\mathscr{B}_p(\mu, \mu_0)$ equipped with the strong operator topology is Hausdorff and quasi-complete: i.e. closed bounded subsets are complete. The closed unit sphere of $\mathscr{B}_p(\mu, \mu_0)$ in the uniform operator topology is closed and bounded in the strong operator topology. Therefore, it is complete.

If a net in $\mathcal{O}^1$ converges $\mathscr{U}_p$ to some $T \in \mathcal{O}$, then it converges $\mathscr{F}$ to T. But this implies $T \in \mathcal{O}^1$ by Theorem 4. Thus, for $1 \leqq p \leqq \infty$, $\mathcal{O}^1$ is closed and convex in $\mathcal{O}(\mu, \mu_0)$ with the $\mathscr{U}_p$ topology.

To prove $\mathcal{O}^1$ is complete, it suffices to show that $\tau^p(\mathcal{O}^1)$ is a closed subset of the closed unit sphere in $\mathscr{B}_p(\mu, \mu_0)$ with respect to the strong operator topology. If a net $\{\tau^p(T_\alpha)\}_{\alpha \in A} \subset \tau^p(\mathcal{O}^1)$ converges strongly to $S \in \mathscr{B}_p(\mu, \mu_0)$, then $\{T_\alpha\}_{\alpha \in A}$ is Cauchy in the $\mathscr{F}$ topology and so converges to some $T \in \mathcal{O}^1$. But then,

$$\langle Sf, g\rangle = \langle Tf, g\rangle \tag{5.33}$$

for $f \in L_\infty(\mu) \cap L_1(\mu)$ and $g \in L_\infty(\mu_0) \cap L_1(\mu_0)$. Therefore, $Sf = Tf$ for f contained in a dense set in $L_p(\mu)$, $1 \leqq p < \infty$. $S = \tau^p(T)$. Q.E.D.

PROPOSITION 7. *On $\mathcal{O}^1$, the $\mathscr{U}_p$ topology is stronger than the $\mathscr{U}_{p'}$ topology for $1 \leqq p \leqq p' < \infty$. If (S_0, Σ_0, μ_0) is a finite measure space, then for $1 \leqq p < \infty$ the $\mathscr{U}_p$ topologies on $\mathcal{O}^1$ are equivalent. In this case, a net $\{T_\alpha\}_{\alpha \in A} \subset \mathcal{O}^1$ converges in the $\mathscr{U}_p$ topology for $1 \leqq p < \infty$ to some $T \in \mathcal{O}^1$ if and only if $\{T_\alpha f\}_{\alpha \in A}$ is Cauchy in measure for each f contained in $L_\infty(\mu) \cap L_1(\mu)$.*

Proof. Let p and p' be as in the first statement of the proposition. Suppose $T_\alpha \to T$ in the $\mathscr{U}_p$ topology. Then $T_\alpha f \to Tf$ in $L_p(\mu_0)$ for each $f \in L_\infty(\mu) \cap L_1(\mu)$. If $\{T_\alpha f\}_{\alpha \in A}$ did not converge to Tf in $L_{p'}(\mu_0)$, then there exists $\varepsilon > 0$ and a subnet $\{T_{\alpha_j} f\}_{j \in J}$ such that for all $j \in J$, $\|T_{\alpha_j} f - Tf\|_{p'} > \varepsilon$. This subnet, of course, converges to Tf in $L_p(\mu_0)$. Because the topology of $L_p(\mu_0)$ is metric, there is a sequence $\{T_{\gamma_n} f\}_{n=1}^{\infty}$ converging to Tf in $L_p(\mu_0)$ such that $T_{\gamma_n} f \in \{T_{\alpha_j} f\}_{j \in J}$ for $n = 1, 2, \ldots$. If it could be shown that $T_{\gamma_n} f \to Tf$ in $L_{p'}(\mu_0)$, a contradiction would be reached proving that $T_\alpha f \to Tf$ in $L_{p'}(\mu_0)$ for $f \in L_\infty(\mu) \cap L_1(\mu)$. The Banach-Steinhaus theorem would then imply that $T_\alpha \to T$ in the $\mathscr{U}_{p'}$ topology.

Noting that for any $E_0 \in \Sigma_0$,

$$\int_{E_0} |T_{\gamma_n} f|^{p'} \mu_0(ds_0) \leqq \|f\|_\infty^{p'-p} \int_{E_0} |T_{\gamma_n} f|^p \mu_0(ds_0) \tag{5.34}$$

for $n=1, 2, \ldots$, $T_{\gamma_n} f \to Tf$ in $L_{p'}(\mu_0)$ by 6 Theorem [5, p. 122].

For $1 \leqq p < \infty$, $T_\alpha \to T$ in the $\mathscr{U}_p$ topology implies $T_\alpha f \to Tf$ in μ_0-measure for $f \in L_\infty(\mu) \cap L_1(\mu)$. If (S_0, Σ_0, μ_0) is a finite measure space and $T_\alpha f \to Tf$ in μ_0-measure for $f \in L_\infty(\mu) \cap L_1(\mu)$, then the inequality

$$\int_{E_0} |T_\alpha f|^p \mu_0(ds_0) \leqq \|f\|_\infty^p \mu_0(E_0) \tag{5.35}$$

and an argument entirely analogous to the preceding one implies $T_\alpha f \to Tf$ in $L_p(\mu_0)$ for $1 \leqq p < \infty$. Hence, $T_\alpha \to T$ in the $\mathscr{U}_p$ topology for $1 \leqq p < \infty$. Thus, the $\mathscr{U}_p$ topologies on $\mathcal{O}^1$ are equivalent for $1 \leqq p < \infty$ when (S_0, Σ_0, μ_0) is a finite measure space. They are, in fact, all equivalent to convergence in measure on $L_\infty(\mu) \cap L_1(\mu)$. The last assertion follows from the completeness of $\mathcal{O}^1$ in the $\mathscr{U}_p$ topologies $1 \leqq p < \infty$. Q.E.D.

6. **Lattice properties.** We already know that $\mathcal{O}(\mu, \mu_0)$ is stable under the mapping $T = j \circ T \circ j$, where j is complex conjugation of functions. $T \in \mathcal{O}(\mu, \mu_0)$ can be written in the form $T = U + iV$, where U and V are elements of $\mathcal{O}(\mu, \mu_0)$ which map real functions into real functions. Explicitly,

$$U = (T + j \circ T \circ j)/2 \tag{6.1}$$

$$V = (T - j \circ T \circ j)/2i. \tag{6.2}$$

Real operators are entirely determined by their restrictions to the spaces L_p^R, $1 \leqq p \leqq \infty$. Let the real operators in $\mathcal{O}(\mu, \mu_0)$ be denoted $\mathcal{O}^R(\mu, \mu_0)$. Then $\mathcal{O}^R(\mu, \mu_0)$ inherits a natural partial order from the spaces $\mathscr{B}^R(\mu, \mu_0)$.

THEOREM 5. *$\mathcal{O}^R(\mu, \mu_0)$ with the natural partial order is a complete Banach lattice.*

Proof. If $0 \leqq f \in L_p(\mu)$ for some fixed p, $1 \leqq p \leqq \infty$, then $T_1 \geqq T_2$ implies $T_1 f \geqq T_2 f$. Because the formula for computing the infimum and supremum of an order-bounded family of mappings is consistent in all the L_p-spaces (see 26 Theorem [5, p. 304]), $\mathcal{O}^R(\mu, \mu_0)$ inherits lattice completeness from the spaces $\mathscr{B}_p^R(\mu, \mu_0)$, $1 \leqq p \leqq \infty$. Q.E.D.

Let the lattice operations supremum and infimum be denoted as usual by "$\vee$" and "$\wedge$" respectively. Each $U \in \mathcal{O}^R(\mu, \mu_0)$ has a unique (Jordan) decomposition

$$U = P_1^+ - P_1^- \tag{6.3}$$

into positive elements

$$P_1^+ = U \vee 0 \tag{6.4}$$

and

$$P_1^- = -(U \wedge 0). \tag{6.5}$$

P_1^+ and P_1^- are disjoint elements of the lattice in the sense

$$P_1^+ \wedge P_1^- = 0. \tag{6.6}$$

If

$$U = Q_1^+ - Q_1^- \tag{6.7}$$

was another such decomposition into positive elements

$$P_1^+ = U \vee 0 = (Q_1^+ - Q_1^-) \vee 0 \leqq Q_1^+ \vee 0 \leqq Q_1^+. \tag{6.8}$$

Similarly,

$$P_1^- \leqq Q_1^-, \tag{6.9}$$

proving the minimality of the Jordan decomposition. Let

$$V = P_2^+ - P_2^- \tag{6.10}$$

be the Jordan decomposition of $V \in \mathcal{O}^R(\mu, \mu_0)$. Then

$$T = U + iV = P_1^+ - P_1^- + iP_2^+ - iP_2^- \tag{6.11}$$

yields a four-parts decomposition for each $T \in \mathcal{O}(\mu, \mu_0)$.

7. **Representation of $\mathcal{O}(\mu, \mu_0)$ as countably additive set functions defined in a certain product space.** The Borel sets of a locally compact Hausdorff topological space S are the members of the smallest σ-algebra containing the open sets of S. A countably additive measure μ defined on a σ-algebra Σ which contains the Borel sets of S is said to have the approximation property if μ is finite on compact sets; for $E \in \Sigma$,

$$\mu(E) = \inf\{\mu(V) \mid E \subset V, V \text{ open}\}; \tag{7.1}$$

and, for F open or of finite measure,

$$\mu(F) = \sup\{\mu(K) \mid K \subset F, K \text{ compact}\}. \tag{7.2}$$

A linear functional l on $C_K(S)$, the continuous functions having compact support, is called positive if $f \geqq 0, f \in C_K(S)$ implies $l(f) \geqq 0$.

The following result appears to be the final form of a theorem first proved in a special case by F. Riesz in 1909: To each positive linear functional l on $C_K(S)$ there corresponds a unique positive measure μ having the approximation property and defined on a σ-algebra Σ containing the Borel sets of S and completed with respect to μ-null sets. μ represents l in the sense that

$$l(f) = \int f(s)\mu(ds) \tag{7.3}$$

for all $f \in C_K(S)$. This statement will be called the Riesz representation theorem (in spite of the fact that many other mathematicians had a hand in the development of the present form of the theorem). A detailed proof of this theorem is given in [11].

If E is a set of finite measure in Σ, then (7.1) and (7.2) imply that there exist sequences $\{V_n\}_{n=1}^{\infty}$ of open sets and $\{K_m\}_{m=1}^{\infty}$ compact sets such that

$$\bigcap_{n=1}^{\infty} V_n \supset E \supset \bigcup_{m=1}^{\infty} K_m, \tag{7.4}$$

$$\mu\left(\bigcap_{n=1}^{\infty} V_n\right) = \mu(E) = \mu\left(\bigcup_{m=1}^{\infty} K_m\right), \tag{7.5}$$

and

$$\mu\left(\bigcap_{n=1}^{\infty} V_n - \bigcup_{m=1}^{\infty} K_m\right) = 0. \tag{7.6}$$

Thus, each finite set in Σ is the union of a Borel set and a subset of a Borel set of μ-measure zero. Thus, all finite sets, hence all σ-finite sets, in Σ are contained in the μ-completion of the Borel sets. On the other hand, Σ is complete with respect to μ and contains all Borel sets. Hence, the μ-completion of the Borel sets is contained in Σ. In particular, when S is σ-finite, then Σ is exactly the μ-completion of the Borel sets.

If (S, Σ, μ), the measure space described in the Riesz representation theorem, is σ-finite and (S, Σ', μ') is another measure space having the properties described in the theorem and inducing the same linear functional l by means of an equation like (7.3), then $\mu = \mu'$ and $\Sigma = \Sigma'$. For, μ and μ' are determined by their values on compact sets. By an argument given in [11, p. 41], $\mu(K) = \mu'(K)$ for all K compact. Therefore, $\mu = \mu'$ on $\Sigma \cap \Sigma'$, a μ-complete σ-algebra which contains the Borel sets and on which μ has the approximation property. It follows that $\Sigma' = \Sigma$, the μ-completion of the Borel sets.

The following remark is of a more technical nature and will be useful later in this section. A determining system for a σ-finite positive linear functional on $C_K(S)$ is a pair (Λ, λ) such that λ is a positive, countably additive set function on the conditional σ-ring Λ: i.e., a ring of subsets of S such that if $\{E_n\}_{n=1}^{\infty} \subset \Lambda$, $E_n \subset E \in \Lambda$ for $n = 1, 2, \ldots$, then $\bigcup_{n=1}^{\infty} E_n \in \Lambda$. The system (Λ, λ) possesses, by definition, the following additional properties (some of which are redundant): (a) Λ contains the compact subsets of S; (b) if $E \in \Lambda$, then $\lambda(E) < \infty$; (c) the intersection of a Borel set and a set in Λ is in Λ; (d) there exists a disjoint sequence $\{E_n\}_{n=1}^{\infty} \subset \Lambda$ such that $S = \bigcup_{n=1}^{\infty} E_n$; (e) if $E \in \Lambda$, then

$$\lambda(E) = \sup\{\lambda(K) \mid K \subset E, K \text{ compact}\}, \tag{7.7}$$

and

$$\lambda(E) = \inf\{\lambda(V) \mid E \subset V, V \text{ open}, V \in \Lambda\}. \tag{7.8}$$

This terminology is justified by the following proposition.

PROPOSITION 8. *Let* (Λ, λ) *be a determining system for a* σ*-finite positive linear functional on* $C_K(S)$. (Λ, λ) *has an extension to a* σ*-finite countably additive measure space* (S, Π_0, λ_0) *which is unique with respect to the properties*

(1) λ_0 *is a positive countably additive measure on* Π_0;

(2) Π_0 *contains the Borel sets and is completed with respect to* λ_0;

(3) λ_0 *has the approximation property on* Π_0.

Then it follows that (S, Π_0, λ_0) *is the representing measure space of a unique positive linear functional on* $C_K(S)$ *determined by* (Λ, λ) *and defined by*

$$\hbar_0(f) = \int f\lambda_0(ds), \qquad f \in C_K(S). \tag{7.9}$$

Π_0 *is the completion of the Borel sets with respect to* λ_0 *restricted to the Borel sets.*

Proof. If (S, Π_0, λ_0) is any extension of (Λ, λ) having properties (1), (2) and (3), then it is necessarily σ-finite and positive. It then follows that (S, Π_0, λ_0) is the representing measure space of $\hbar_0$. If (S, Π_1, λ_1) is another extension having properties (1), (2), and (3), and $\hbar_1$ is the corresponding positive linear functional, then $(S, \Pi_0, \lambda_0) = (S, \Pi_1, \lambda_1)$. The proof is exactly like the preceding remark except that λ_0 and λ_1 agree on compact sets by assumption. It follows that $\hbar_1 = \hbar_0$, and (S, Π_0, λ_0) is unique with respect to properties (1), (2), and (3).

The required extension exists. Λ is a ring of sets on which λ is finite positive, and countably additive. Therefore, λ has a unique extension to a σ-finite positive countably additive set function on Π_0', the smallest σ-ring over Λ. But $S \in \Pi_0'$ by (d) in the definition of determining system. Therefore, (S, Π_0', λ) is a positive σ-finite measure space. Let (S, Π_0, λ_0) be its completion. It remains to show that Π_0 contains the open sets of S, which are the generators of the Borel σ-algebra, and λ_0 has the approximation property on Π_0.

$S = \bigcup_{i=1}^{\infty} E_i$, $E_i \cap E_j = \varnothing$ for $i \neq j$, and $E_i \in \Lambda$ for $i = 1, 2, \ldots$. Hence, there exist $\{V_i\}_{i=1}^{\infty} \subset \Lambda$ such that $V_i \supset E_i$ and V_i is open for $i = 1, 2, \ldots$. If V is an arbitrary open set in S, then $V \cap V_i \in \Lambda$ for $i = 1, 2, \ldots$ by property (c) of a determining system. Therefore,

$$V = V \cap S = V \cap \bigcup_{i=1}^{\infty} V_i = \bigcup_{i=1}^{\infty} (V \cap V_i) \tag{7.10}$$

and $V \in \Pi_0' \subset \Pi_0$. If F is a Borel set, then

$$F = \bigcup_{i=1}^{\infty} (F \cap E_i). \tag{7.11}$$

Each $F \cap E_i \in \Lambda$ for $i = 1, 2, \ldots$. Now, using the approximation property on Λ, it is easy to show that λ has the approximation property on the Borel sets. Let $(S, \Pi_0'', \lambda_0'')$ be the measure space such that Π_0'' is the completion of λ_0 restricted to the Borel sets. $(S, \Pi_0'', \lambda_0'')$ has the properties that Π_0'' is complete and contained in Π_0; λ_0'' has the approximation property on Π_0'' and $\lambda_0 = \lambda_0''$ on Π_0''. Further, $\Lambda \subset \Pi_0''$, since every set in Λ is the union of a Borel set and a subset of a Borel set of λ-measure zero. Therefore, $\Pi_0' \subset \Pi_0''$, implying that the completion $\Pi_0 \subset \Pi_0''$. Therefore, $\Pi_0 = \Pi_0''$. Q.E.D.

Theorem 6 of this section says, roughly, that if (S, Σ, μ) and (S_0, Σ_0, μ_0) are σ-finite positive measure spaces constructed on locally compact Hausdorff topological spaces having the properties of completeness, containment of the Borel sets, and approximation, then each Banach positive operator $T \in \mathcal{O}(\mu, \mu_0)$ is represented by a unique σ-finite positive measure on $S \times S_0$. The usefulness of this theorem can be extended by a result due to Segal [12]: If (M, Θ, η) is any σ-finite positive measure space, then there exists a perfect measure space (S, Σ, μ) such that the measure algebra of (M, Θ, η) is measure-preservingly σ-isomorphic to the measure algebra of (S, Σ, μ). (For the proof, see Theorem 6.1 in [12].) The corresponding perfect measure space satisfies the requirements for Theorem 6 of this section.

If λ is an additive set function defined on a product space $(S \times S_0, \Sigma \times \Sigma_0)$, then $\lambda^{(1)}$ is the projection of λ on the first coordinate space and is defined by

$$\lambda^{(1)}(E) = \lambda(E \times S_0), \qquad E \in \Sigma. \tag{7.12}$$

$\lambda^{(2)}$ is defined similarly as projection on the second coordinate space.

In order to avoid cumbersome repetitions in the following theorem, let us call a σ-finite positive measure space (S, Σ, μ) such that S is a locally compact Hausdorff topological space and μ is countably additive and has the approximation property on a σ-algebra Σ which contains the Borel sets of S and is completed with respect to μ, a σ-finite regular measure space. This terminology is not entirely at variance with the customary usage; for, in the course of the proof of Proposition 8, it was shown that each set in Σ was regular.

THEOREM 6. *Let (S, Σ, μ) and (S_0, Σ_0, μ_0) be σ-finite regular measure spaces. If T is a Banach positive operator in $\mathcal{O}(\mu, \mu_0)$, then there exists a unique σ-finite regular measure space $(S \times S_0, \Pi, \lambda_T)$ such that*

$$\lambda_T^{(1)} \leqq c_1 \mu, \qquad c_1 \text{ a nonnegative scalar}; \tag{7.13}$$

$$\lambda_T^{(2)} \leqq c_2 \mu_0, \qquad c_2 \text{ a nonnegative scalar}; \tag{7.14}$$

$$\int Tf \cdot g \mu_0(ds_0) = \int f(s) g(s_0) \lambda_T(ds \times ds_0), \tag{7.15}$$

$f \in L_1(\mu)$, $g \in L_\infty(\mu_0)$.

Conversely, if $(S \times S_0, \Pi, \lambda_T)$ is a σ-finite regular measure space and is such that (7.13) *and* (7.14) *are satisfied then* (7.15) *defines a unique Banach positive operator $T \in \mathcal{O}(\mu, \mu_0)$ such that*

$$\|T\|_{\mathcal{O}(\mu,\mu_0)} \leqq \text{Max}\,\{c_1, c_2\}. \tag{7.16}$$

Proof. If T is Banach positive and an element of $\mathcal{O}(\mu, \mu_0)$, then T induces a unique continuous linear map of $L_1(\mu)$ into $C_0(S_0)^*$, the dual of the continuous functions on S_0 vanishing at infinity. ($(C_0(S_0))^*$ is exactly the set of bounded linear functionals on $C_K(S_0)$.) For if $0 \leqq f \in L_1(\mu)$ and $g \in C_K(S_0)$,

$$\mathscr{k}(g) = \int Tf \cdot g\mu_0(ds_0) \tag{7.17}$$

defines a positive bounded linear functional on $C_K(S_0)$. Using the four-parts decomposition of elements of $L_1(\mu)$, this obvious correspondence between Tf and $\mathscr{k}$ can be extended to all of $L_1(\mu)$. The mapping $f \to \mathscr{k}$ is linear and takes positive functions into positive functionals.

The Riesz representation theorem implies that if $0 \leqq f \in L_1(\mu)$ and $\mathscr{k}$ is the functional induced by Tf, then $\mathscr{k}$ is represented by a unique finite regular measure space based on S_0. However, φ, the indefinite integral of Tf with respect to μ_0, is a uniquely determined finite positive measure on Σ_0 which is absolutely continuous with respect to μ_0,

$$\|\varphi\|_{\mathrm{ba}(S_0,\Sigma_0,\mu_0)} = \|Tf\|_1. \tag{7.18}$$

Because of absolute continuity, φ inherits the approximation property on Σ_0 from μ_0. Let $(S_0, \Sigma_1, \bar{\varphi})$ be the completion of φ on Σ_0: it is a finite regular measure space. But the positive linear functional which $\bar{\varphi}$ induces on $C_K(S_0)$ by integration is, by definition, $\mathscr{k}$ induced by Tf. Therefore,

$$\|\mathscr{k}\|_{C_0(S_0)^*} = \|\varphi\|_{\mathrm{ba}(S_0,\Sigma_0,\mu_0)} = \|Tf\|_1. \tag{7.19}$$

Thus, the mapping $f \to \mathscr{k}$ is continuous.

By Proposition 8 [6, p. 62], there exists a unique linear functional l on $C_K(S \times S_0)$ such that if $f \in L_1(\mu)$ and $g \in C_K(S_0)$ then

$$l(f \cdot g) = \int Tf \cdot g\mu_0(ds_0). \tag{7.20}$$

Since the functional l is positive on the linear span of $C_K(S) \times C_K(S_0)$, a "positively rich" subspace of $C_K(S \times S_0)$ in the terminology of N. Bourbaki, l is positive on $C_K(S \times S_0)$. (See Chapter III, §2, no. 5 and §5, no. 1 in [2] for this circle of ideas.) Therefore, there exists a positive measure λ_T and a σ-algebra Π representing l on $C_K(S \times S_0)$. (7.20) then becomes (7.15) in the special case $g \in C_K(S_0)$. The proposition cited in [6] also shows that (7.13) holds, proving that λ_T is σ-finite. Thus, $(S \times S_0, \Pi, \lambda_T)$ is a uniquely determined σ-finite regular measure space.

Exactly the same considerations applied to the operator T^* show the existence of a uniquely determined σ-finite regular measure space $(S_0 \times S, \Pi^*, \lambda_{T^*})$ such that

$$\lambda_{T^*}^{(1)} \leqq c_1^* \mu_0, \qquad c_1^* \text{ a nonnegative scalar}; \tag{7.21}$$

$$\int T^* f_1 \cdot g_1 \mu(ds) = \int f_1(s_0) g_1(s) \lambda_{T^*}(ds_0 \times ds), \tag{7.22}$$

$f_1 \in L_1(\mu_0)$, $g_1 \in L_\infty(\mu)$.

Let $\tilde{\lambda}_{T^*}$ and $\tilde{\Pi}^*$ be the measure and σ-algebra obtained in $S\times S_0$ from λ_{T^*} and Π^* by the transformation $(s_0, s)\to(s, s_0)$. If $f\in C_K(S)$ and $g\in C_K(S_0)$,

$$
(7.23)\qquad \begin{aligned}\int f(s)g(s_0)\lambda_T(ds\times ds_0) &= \int Tf\cdot g\mu_0(ds_0) = \int f\cdot T^*g\mu(ds)\\ &= \int g(s_0)f(s)\lambda_{T^*}(ds_0\times ds) = \int f(s)g(s_0)\tilde{\lambda}_{T^*}(ds\times ds_0).\end{aligned}
$$

Thus, the linear functionals induced by λ_T and $\tilde{\lambda}_{T^*}$ coincide on $C_K(S)\times C_K(S_0)$. Therefore, they agree on $C_K(S\times S_0)$. Since they are both σ-finite regular measures, $\Pi=\tilde{\Pi}^*$ and $\lambda_T=\tilde{\lambda}_{T^*}$ on Π. (7.21) implies that (7.14) holds for λ_T with $c_2=c_1^*$. It remains to show that (7.15) holds for $g\in L_\infty(\mu_0)$. This, however, is a straightforward exercise in measure theory and its proof will be omitted.

Conversely, suppose $(S\times S_0, \Pi, \lambda_T)$ is given. Using (7.13), it is easy to show that the integrals on the right-hand side of (7.15) are finite. Further, (7.15) shows that T, if it exists, is linear and positive. For fixed $f\in L_1(\mu)$, the inequality

$$
(7.24)\qquad \left|\int f(s)g(s_0)\lambda_T(ds\times ds_0)\right| \leqq c_1\|f\|_1\cdot\|g\|_\infty,
$$

which can be deduced from (7.13), implies that f defines a unique continuous linear functional on $L_\infty(\mu_0)$. The countable additivity of λ_T permits this functional to be realized as a finite, countably additive, μ_0-continuous set function on Σ_0. The Radon-Nikodym derivative of this set function with respect to μ_0 defines a unique member of $L_1(\mu_0)$ which is, by definition, Tf. (7.24) implies $\|T\|_1\leqq c_1$; and, hence, $\|T^*\|_\infty\leqq c_1$. If $g_1\in L_\infty(\mu)$ and $f_1\in L_1(\mu_0)$, a similar argument shows that

$$
(7.25)\qquad \left|\int f_1(s_0)g_1(s)\lambda_{T^*}(ds_0\times ds)\right| \leqq c_1^*\|f_1\|_1\cdot\|g_1\|_\infty.
$$

But, $c_1^*=c_2$, and

$$
(7.26)\qquad \begin{aligned}\left|\int f_1(s_0)g_1(s)\lambda_{T^*}(ds_0\times ds)\right| &= \left|\int g_1(s)f_1(s_0)\tilde{\lambda}_{T^*}(ds\times ds_0)\right|\\ &= \left|\int g_1(s)f_1(s_0)\lambda_T(ds\times ds_0)\right| \leqq c_2\|f_1\|_1\,\|g_1\|_\infty.\end{aligned}
$$

Thus, T^* has a consistent extension to $L_1(\mu_0)$, and $\|T^*\|_1\leqq c_2$. Therefore,

$$
(7.27)\qquad \|T\|_{\mathcal{O}(\mu,\mu_0)} = \|T^*\|_{\mathcal{O}(\mu_0,\mu)} \leqq \operatorname{Max}\{c_1, c_2\}
$$

by the Riesz convexity theorem. Q.E.D.

COROLLARY 1. *λ_{T^*} is obtained from λ_T by the transformation $(s, s_0)\to(s_0, s)$.*

Proof. This fact was obtained in the course of the proof of the theorem.

COROLLARY 2. *If T is a Banach positive operator in $\mathcal{O}(\mu, \mu_0)$, λ_T is its corresponding product measure, and $f \in L_1(\mu)$, then ρ_f is the μ_0-continuous, finite, countably additive set function on Σ_0 defined by*

$$\rho_f(E_0) = \int f(s)\chi_{E_0}(s_0)\lambda_T(ds \times ds_0). \tag{7.28}$$

And,

$$Tf = d\rho_f/d\mu_0. \tag{7.29}$$

Proof. This fact, too, was obtained in the course of the proof of the theorem.

COROLLARY 3. *If λ and ω are two measures on $S \times S_0$ satisfying the hypotheses of the theorem and*

$$\lambda(E \times E_0) = \omega(E \times E_0) \tag{7.30}$$

for all $E \in \Sigma$ and $E_0 \in \Sigma_0$ such that $\mu(E) < \infty$ and $\mu_0(E_0) < \infty$, then $\lambda = \omega$ and the two measures have the same domain of definition.

Proof. Let T be defined by λ and S by ω. S and T are contained in some closed bounded sphere of $\mathcal{O}(\mu, \mu_0)$ centered at zero. The $\mathcal{F}$ topology on this sphere is Hausdorff and completely determined by linear functionals

$$\eta(B) = \int B\chi_E \cdot \chi_{E_0}\mu_0(ds_0), \tag{7.31}$$

$B \in \mathcal{O}(\mu, \mu_0)$, $E \in \Sigma$ and $E_0 \in \Sigma_0$ such that $\mu(E) < \infty$ and $\mu_0(E_0) < \infty$. But $\eta(T) = \eta(S)$ for all such functionals. Therefore, $S = T$ and $\lambda = \omega$ by the uniqueness of the representing measure. Q.E.D.

In §6, it was shown that a general operator $T \in \mathcal{O}(\mu, \mu_0)$ has a representation, necessarily unique, as

$$T = U + iV, \tag{7.32}$$

where U and V are elements of $\mathcal{O}(\mu, \mu_0)$ which map real functions into real functions. These real operators were shown to have a unique Jordan decomposition into positive elements of $\mathcal{O}(\mu, \mu_0)$,

$$U = U_1 - U_2, \tag{7.33}$$

$$V = V_3 - V_4. \tag{7.34}$$

Thus, T has a unique four-parts decomposition into

$$T = U_1 - U_2 + i(V_3 - V_4). \tag{7.35}$$

According to Theorem 6, let λ_1 on Π_1 correspond to U_1, λ_2 on Π_2 to U_2, λ_3 on Π_3 to V_3, and λ_4 on Π_4 to V_4. Let

$$\Lambda = \bigcap_{i=1}^{4} \{F \mid F \in \Pi_i, \lambda_i(F) < \infty\}. \tag{7.36}$$

Then, Λ has properties (a) through (d) listed in the definition of a determining system and is a conditional σ-ring. If

$$\lambda_T = \lambda_1 - \lambda_2 + i\lambda_3 - i\lambda_4, \tag{7.37}$$

then λ_T is a finite-valued, countably additive set function on Λ. Λ and λ_T are uniquely determined with respect to the process described above.

Define

$$\nu(\lambda_T, E) = \sup_{\mathscr{P}} \sum_{i=1}^{n} |\lambda_T(E_i)|, \tag{7.38}$$

where $\mathscr{P}$ is the set of all finite partitions $\{E_1, E_2, \ldots, E_n\}$ of E such that $E_i \in \Lambda$ for $i=1, 2, \ldots, n$, where n is any positive integer. Since

$$|\lambda_T(E)| \leqq \nu(\lambda_T, E) \leqq \sum_{i=1}^{4} \lambda_i(E), \tag{7.39}$$

it is easy to show, first, that $\nu(\lambda_T, \cdot)$ is additive and finite-valued on Λ (see 6 Lemma [5, p. 98]); second, that $\nu(\lambda_T, \cdot)$ is countably additive on Λ; and, third, that $\nu(\lambda_T, \cdot)$ has the approximation property (e) on Λ listed in the definition of a determining system. Therefore, $(\Lambda, \nu(\lambda_T))$ is a determining system and determines a unique σ-finite regular measure space on $S \times S_0$. Inequality (7.39) implies that this extended measure, also denoted $\nu(\lambda_T)$, satisfies (7.13) and (7.14). According to Theorem 6, the measure defines a unique Banach positive operator $Q_T \in \mathcal{O}(\mu, \mu_0)$.

THEOREM 7. *If $T \in \mathcal{O}(\mu, \mu_0)$, then $Q_T = P_T$. Equivalently, $\nu(\lambda_T) = \lambda_{P_T}$ and both σ-finite regular measures have the same domain Π_0, the Borel sets of $S \times S_0$ completed with respect to $\nu(\lambda_T)$.*

Proof. It suffices to show that $\nu(\lambda_T) = \lambda_{P_T}$ on sets of the form $E \times E_0$, where $\mu(E) < \infty$ and $\mu_0(E_0) < \infty$, by Corollary 3 to Theorem 6.

LEMMA 1. *If $T \in \mathcal{O}(\mu, \mu_0)$ and P_T is its positive, then for each Borel set $H \in \Lambda$*

$$|\lambda_T(H)| \leqq \lambda_{P_T}(H). \tag{7.40}$$

Proof. Because

$$|\lambda_T(H)| \leqq \nu(\lambda_T, H), \tag{7.41}$$

given $\varepsilon > 0$, there exist K, a compact set in $S \times S_0$, and W, an open set in $S \times S_0$ contained in Λ, such that $K \subset H \subset W$ and

$$|\lambda_T(H) - \lambda_T(K)| < \varepsilon/4, \tag{7.42}$$

$$|\lambda_{P_T}(H) - \lambda_{P_T}(K)| < \varepsilon/4, \tag{7.43}$$

$$|\lambda_T(W) - \lambda_T(H)| < \varepsilon/4, \tag{7.44}$$

$$|\lambda_{P_T}(W) - \lambda_{P_T}(H)| < \varepsilon/4, \tag{7.45}$$

$$|\nu(\lambda_T, H) - \nu(\lambda_T, K)| < \varepsilon/4, \tag{7.46}$$

$$|\nu(\lambda_T, W) - \nu(\lambda_T, H)| < \varepsilon/4. \tag{7.47}$$

For each $(s, s_0) \in K$ there exists $V \times V^0$ such that

$$(s, s_0) \in V \times V^0 \subset W, \tag{7.48}$$

where $s \in V$, $s_0 \in V^0$, V is open in S, and V^0 is open in S_0. $V \times V^0$ is a Borel set of $S \times S_0$ and is contained in Λ. A finite number, $\{V_i \times V_i^0\}_{i=1}^n$ cover K and

$$\bigcup_{i=1}^{n} (V_i \times V_i^0) \subset W. \tag{7.49}$$

Let $\{D_j\}_{j=1}^m$ be the disjoint refinement of this open cover. (D_j is no longer open, but it is still a member of Λ.) Each D_j is of the form $W_j \times W_j^0$. Then,

$$\left|\lambda_T\left(H - \bigcup_{j=1}^{m} D_j\right)\right| \leqq \nu(\lambda_T, W-K) \leqq \nu(\lambda_T, W-H) + \nu(\lambda_T, H-K) \leqq 2\varepsilon/4. \tag{7.50}$$

Similarly,

$$\left|\lambda_{P_T}\left(H - \bigcup_{j=1}^{m} D_j\right)\right| \leqq 2\varepsilon/4. \tag{7.51}$$

But

$$\begin{aligned} |\lambda_T(H)| &\leqq \left|\sum_{j=1}^{m} \lambda_T(D_j)\right| + 2\varepsilon/4 \\ &\leqq \sum_{j=1}^{m} |\langle T\chi_{W_j}, \chi_{W_j^0}\rangle| + 2\varepsilon/4 \\ &\leqq \sum_{j=1}^{n} \langle P_T\chi_{W_j}, \chi_{W_j^0}\rangle + 2\varepsilon/4 \\ &\leqq \lambda_{P_T}\left(\bigcup_{j=1}^{m} D_j\right) + 2\varepsilon/4. \end{aligned} \tag{7.52}$$

However,

$$\lambda_{P_T}\left(\bigcup_{j=1}^{m} D_j\right) \leqq \lambda_{P_T}(H) + 2\varepsilon/4. \tag{7.53}$$

Hence,

$$|\lambda_T(H)| \leqq \lambda_{P_T}(H) + \varepsilon. \tag{7.54}$$

Since ε is arbitrary,

$$|\lambda_T(H)| \leqq \lambda_{P_T}(H). \tag{7.55}$$

Q.E.D.

There exists a disjoint partition $\{H_k\}_{k=1}^l$ of $E \times E_0$ such that for a given $\varepsilon > 0$

$$\nu(\lambda_T, E \times E_0) - \varepsilon \leqq \sum_{k=1}^{l} |\lambda_T(H_k)|. \tag{7.56}$$

$$H_k = H_k' \cup N_k, \qquad k = 1, 2, \dots, l, \tag{7.57}$$

where H_k' is a Borel set in Λ such that $\lambda_T(H_k') = \lambda_T(H_k)$ and $\lambda_T(N_k) = 0$. Hence,

$$\nu(\lambda_T, E \times E_0) - \varepsilon \leqq \sum_{k=1}^{l} |\lambda_T(H_k')| \leqq \sum_{k=1}^{l} \lambda_{P_T}(H_k) \leqq \lambda_{P_T}(E \times E_0). \tag{7.58}$$

Because ε was arbitrary,

$$\nu(\lambda_T, E\times E_0) \leqq \lambda_{P_T}(E\times E_0). \tag{7.59}$$

To prove the reverse inequality, let $g=\sum_{j=1}^m \beta_j\chi_{G_j}$, $G_j \cap G_k=\varnothing$ for $j\neq k$, and $|g|<\chi_{E_0}$. Then, $|\beta_j|\leqq 1$ for $j=1,\ldots,m$, and

$$\begin{aligned} \left|\int T\chi_E\cdot g\mu_0(ds_0)\right| &\leqq \sum_{j=1}^m |\beta_j| \left|\int T\chi_E\cdot\chi_{G_j}\mu_0(ds_0)\right| \\ &\leqq \sum_{j=1}^m |\lambda_T(E\times G_j)| \leqq \nu(\lambda_T, E\times E_0). \end{aligned} \tag{7.60}$$

Let L' be the set of simple functions in $L_\infty(\mu_0)$.

$$\langle |T\chi_E|, \chi_{E_0}\rangle = \sup_{|g|\leqq\chi_{E_0};\, g\in L'} |\langle T\chi_E, g\rangle| \leqq \nu(\lambda_T, E\times E_0) = \langle Q_T\chi_E, \chi_{E_0}\rangle. \tag{7.61}$$

This implies

$$|T\chi_E| \leqq Q_T\chi_E. \tag{7.62}$$

If $f=\sum_{i=1}^n \alpha_i\chi_{F_i}$, $F_i\cap F_k=\varnothing$ for $i\neq k$, and $|f|\leqq\chi_E$, then $|\alpha_i|\leqq 1$ for $i=1, 2,\ldots, n$; and

$$|Tf| \leqq \sum_{i=1}^n |T\chi_{F_i}| \leqq \sum_{i=1}^n Q_T\chi_{F_i} \leqq Q_T\chi_E. \tag{7.63}$$

Let L be the set of simple functions in $L_\infty(\mu)$. By Proposition 1,

$$P_T\chi_E = \sup_{|f|\leqq\chi_E;\, f\in L} |Tf| \leqq Q_T\chi_E. \tag{7.64}$$

Therefore,

$$\lambda_{P_T}(E\times E_0) = \langle P_T\chi_E, \chi_{E_0}\rangle \leqq \langle Q_T\chi_E, \chi_{E_0}\rangle = \nu(\lambda_T, E\times E_0). \tag{7.65}$$

Combining (7.59) and (7.65),

$$\lambda_{P_T}(E\times E_0) = \nu(\lambda_T, E\times E_0), \tag{7.66}$$

completing the proof of Theorem 7. Q.E.D.

For general σ-finite measure spaces (M, Θ, η) and (M_0, Θ_0, η_0) there exist measure spaces (S, Σ, μ) and (S_0, Σ_0, μ_0) satisfying the hypotheses of Theorem 6 such that the measure algebra of (M, Θ, η) is measure-preservingly isomorphic to the measure algebra of (S, Σ, μ) and (M_0, Θ_0, η_0) is similarly related to (S_0, Σ_0, μ_0). The proof is by means of the isomorphism into the corresponding perfect measure spaces mentioned in the remarks preceding Theorem 6. Thus, if T is a Banach positive operator in $\mathcal{O}(\eta, \eta_0)$, there exists an isometric, order-preserving, adjoint-preserving isomorphism of $\mathcal{O}(\eta, \eta_0)$ onto $\mathcal{O}(\mu, \mu_0)$. Theorem 6 now implies that T is represented by a unique σ-finite regular measure on the product $S\times S_0$.

Theorem 6 as well as some of the results in §5 generalize theorems contained in J. Brown [3]. Brown considers the case of Markov operators on a probability space. He implicitly credits Theorem 6, at least in the special case he discusses, to an unpublished manuscript of J. Lindenstrauss. However, Brown's proof method is quite different from the one utilized here.

Most of the preceding results for σ-finite measure spaces depend on the validity of the Radon-Nikodym theorem, the duality theorems for L_p-spaces, and the lattice properties of the L_p-spaces. These requirements are simultaneously satisfied for a class of measure spaces called localizable. The author conjectures that the relevant results given in §§3 through 7 generalize to localizable measure spaces.

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